Noise in Stokes spectra obtained with XPOL

We start out with the radiometer formula which gives the rms noise σ_{rf} in spectra obtained with a heterodyne receiver of one polarization

$$\sigma_{rf} = \frac{KT_{sys}}{\eta\sqrt{\Delta\nu\ \tau}}$$

As usual, $\Delta\nu$ and τ are the spectrometer noise equivalent bandwidth and τ is the integration time. T_{sys} is the system temperature, while η is the spectrometer efficiency. K characterizes the observing mode.

In XPOL we always observe two orthogonal linear polarizations, H and V. In the standard calibration scheme, these two polarizations are *averaged* to obtain the total power spectrum I. The noise in the I–spectrum is then

$$\sigma_I = \sigma_{RF}/\sqrt{2}$$

The time estimator for EMIR gives in fact the noise for the 2 polarizations averaged, i.e. σ_I.

The standard calibration scheme however assumes that unpolarized sources are observed. The resulting temperature scale, i.e. spectrl *power* is therefore a factor 2 too high. In polarimetry, the noise in the H and $V-$ spectra is therefore

$$\sigma_H = \sigma_V = \sigma_{rf}/2$$

Starting from here, it is easy to see that the noise in the Stokes Q–spectrum is

$$\sigma_Q = \left[\sigma_H^2 + \sigma_V^2\right]^{1/2} = \sigma_{rf}/\sqrt{2} = \sigma_I$$

Similar considerations for the Stokes U and V spectra give the same result:

$$\sigma_U = \sigma_V = \sigma_I$$

We conclude that all four Stokes spectra have the same noise, namely σ_I, which is directly obtained from the IRAM 30m time estimator. Note however the the time estimator gives effective *observing* times which are larger the the integration time τ of the radiometer formula.

The formal errors of the linear and circular degrees of polarization and of the polarization angle are then derived from the noise of the Stokes spectra by error propagation.

Sensitivity to magnetic fields

The normal Zeeman effect
In the presence of a line–of–sight component B_{los} a transition of an atom or molecule splits in the simplest case into the 3 components present in the normal Zeeman effect designated as $\sigma^-, \pi,$ and σ^+. The $\sigma^\pm$ components are offset from the unshifted π–component by $\pm\delta\nu$, Hz, which is proportional to B_{los}, μG :

$$\delta\nu = 1.40 g_L B_{los}$$

where g_L is the effective Landé factor of the transition. It is unity for hydrogen recombination lines, but it can attain different values for molecular transitions involving the spin of the electron, and it is much smaller for non-paramagnetic molecules. The frequency separation between the Zeeman σ^+ and σ^- components may also be described by a "magnetic sensitivity" factor

$$Z = 2.80 g_L \ [\mu\mathrm{Hz/G}]$$

Since the $\sigma^\pm$ components are circularly polarized, the Stokes V-spectrum of a transition attains a characteristic S–shape (see Figure) whose properties we derive here.

The Stokes V–spectrum
Without a field, we have for an unpolarized line

$$R(\nu) = L(\nu) = \frac{1}{2} I(\nu)$$

where $I, R,$ and L denote the total power, and the right and left circularly polarized spectra. Switching on a *weak* field shifts the R and L components by $\pm\delta\nu$ which we take to be much smaller than the Doppler width of the line. For the generally weak fields encountered in radio astronomy, this assumption is always valid for non–masing transitions.

$$\begin{array}{lclcl} R(\nu) & \longrightarrow & R(\nu+\delta\nu) & = & R(\nu) + \frac{dR(\nu)}{d\nu}\delta\nu \\ L(\nu) & \longrightarrow & L(\nu-\delta\nu) & = & L(\nu) - \frac{dL(\nu)}{d\nu}\delta\nu \end{array}$$

The sign of $\delta\nu$ is only indicative here. Its physical meaning is decussed below. The Stokes V–spectrum is then

$$V(\nu) = R(\nu) - L(\nu) = \left[\frac{dR(\nu)}{d\nu} - \frac{dR(\nu)}{d\nu}\right]\delta\nu = \frac{dI(\nu)}{d\nu}\delta\nu$$

where the last equality makes use again of the assumption that $\delta\nu$ is small.

Gaussian and its derivatives
We approximate the shape of a spectral line by a gaussian where A is its amplitude and σ its 1/e–half width. Note that the usual FWHP is $2\sqrt{2\ln 2}\,\sigma$ $(= 2.35\sigma)$. y' and y'' denote the first and second derivative of the line profile y which is a function if x, usually the frequency.

$$
\begin{aligned}
y &= Ae^{-\frac{x^2}{2\sigma^2}} && (1)\\
y' &= -\frac{x}{\sigma^2}Ae^{-\frac{x^2}{2\sigma^2}} && (2)\\
y'' &= -\frac{A}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left[1-\frac{x^2}{\sigma^2}\right] && (3)
\end{aligned}
$$

We see that the derivative y' peaks at $x = \pm\sigma$. These are the turning points of the gaussian where $y'' = 0$.

The S–shaped Zeeman feature

If the line under study can be approximated by a gaussian we can combine eqs. 2 and xx to obtain the spectral shape of the Stokes V–spectrum. The positive and negative peaks of the V–spectrum, V_{max}, occur at frequency offsets of $\delta\nu = \pm\sigma$ and their values are

$$\frac{V_{max}}{I_{max}} = \frac{1}{\sqrt{e}}\frac{\delta\nu}{\sigma}$$

As an example, we consider the H30α recombination line at 231.9 GHz in a source where the line width is 10 $\mathrm{km\,s^{-1}}$ (FWHP) and B_{los} was found to be 30 mG. The Zeeman components are then shifted by $\delta\nu = \pm 42$ kHz, while the 1/e–line half–width is $\sigma = 3.4$ MHz, much larger than $\delta\nu$. The observed S–shaped Zeeman feature has peak values of $V_{max}/I_{max} = \pm 0.012$.

Required observing time

With the value of V_{max}/I_{max} at hand, it is straightforward to estimate the required observing time using the Time Estimator on the IRAM web site. In the example above where the total power of H30α is $I_{max} = 2$ K, we need an rms of 8 mK in the V–spectrum if the two peaks of the V–spectrum are to be detected with S/N = 3. Under average winter conditions and using VESPA with a spctral resolution of 625 kHz, the Time estimator gives 3.1 hours of observing time, including the usual overheads.

In this example, the actual detection of the magnetic field is even better than 3σ because *(i)* the V–spectrum can be smoothed and *(ii)* the spectrum has two peaks. Note however that VESPA is the only backend available for XPOL observations. Its best spectral resolution in polarimetry mode is 40 kHz, barely sufficient for molecular transitions in the very coldest clouds.

The polarity of the V–spectrum (to be verified)

At the 30m telescope the XPOL spectra are calibrated by MIRA. It outputs Stokes V–spectra whose polarity is compatiple with the respective IAU convention (see the background material at the end of this wiki page). What does this imply for the polarity of the S–shaped Zeeman feature ?

We start by recalling the definition of the $\sigma^{\pm}$–components of a Zeeman triplet which is the result of transitions with $\Delta M_J = \pm 1$. As long as the effective g_L of a transition is positive (most frequent case), the $\Delta M_J = +1$ transitions have a higher energy than the $\Delta M_J = -1$ ones, and are therefore shifted by $+\delta\nu$.

In the usual coordinate system where the z–axis is pointing in the same direction as $\vec{B}$, the z–component of the angular momentum J_z changes by

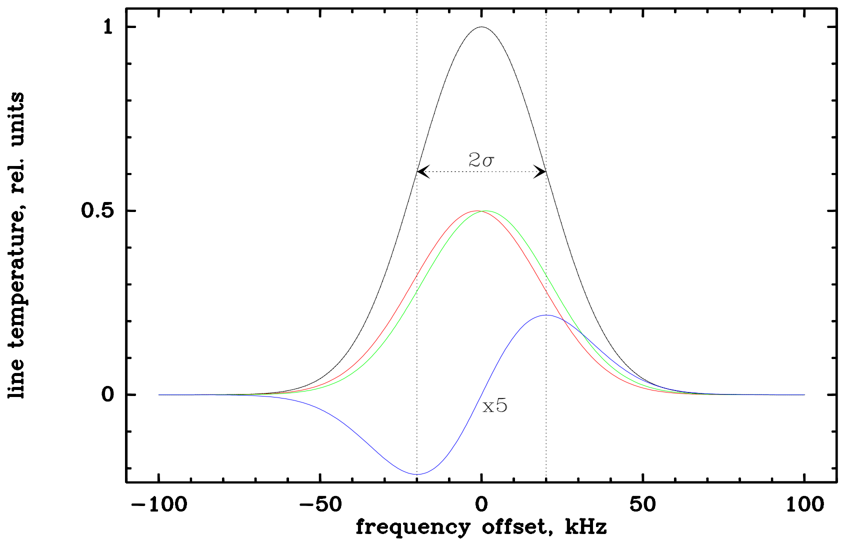

Figure 1: A molecular transition of σ = 20 kHz (FWHP = 0.14 $\mathrm{km\,s^{-1}}$ at 100 GHz). A line–of–sight magnetic field of 1 mG pointing away from the observer shifts the RHC (σ^+) component by +1.4 kHz (green) and the LHC component (red) by −1.4 kHz. The transition is assumed to have an effective Landé factor of $g_L = +1$. The resulting Stokes V–spectrum ($V = RHC - LHC$) is shown amplified by a factor 5.

one unit, $+\hbar$, when the atom or molecule makes a $\Delta M_J = +1$ transition from an upper to a lower level. This represents a loss from the system of one unit of *positive* angular momentum which is carried away by the photon. The angular momentum vector of this photon is therefore parallel to $\vec{B}$.

If this photon is travelling in the $+z$–direction, the projection of its angular momentum vector onto its velocity vector is positive. Such photons are said to have positive helicity, and a beam of such photons is said to be right hand circularly polarized. Here we use the nomenclature of particle physics which is compatiple the IEEE standard used also in radio astronomy (see the background material on the PolarimetryforAstronomers wiki page).

In summary, the Zeeman spectrum of a transition with positive effective Landé factor has its positive peak at a positive frequency offset if the line–of–sight magnetic field points away from the observer.